MULTIPOLE CORRECTIONS TO PERIHELION AND NODE LINE PRECESSION

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In this talk relativistic corrections due to Geroch-Hansen multipoles for perihelion precession and node line precession of orbits in a stationary axially symmetric vacuum spacetime endowed with a plane of symmetry will be shown. Patterns of regularity will be discussed.

1 Introduction

According to Kepler's law, a probe particle orbiting around a central spherical mass describes a trajectory with the shape of an ellipse. When particles are not pointlike or they are not even spherical, distortions appear and orbits are no longer closed, according to Bertrand's theorem. In such cases we talk of precession.

There are many phenomena comprised in this word, the precession of the rotation of the axis or the Earth, the precession of equinoxes... We shall deal with two of them.

First of all, we shall consider perihelion precession. In classical mechanics, it arises from deviations from sphericity, whereas in general relativity it comes up even for spherical bodies. Next we shall take into account the precession of the line of nodes of a nearly equatorial orbit.

2 Mathematical framework

For our purposes, we shall derive a stationary axially symmetric metric with a plane of symmetry. It is well known that Einstein equations in this case reduce to a second order partial differential equation for a complex function, the Ernst potential, ε,[1]

$$\varepsilon_{\rho\rho} + \frac{1}{\rho}\varepsilon_\rho + \varepsilon_{zz} = \frac{2}{\varepsilon + \bar{\varepsilon}}({\varepsilon_\rho}^2 + {\varepsilon_z}^2), \tag{1}$$

$$\varepsilon = f + i\chi, \tag{2}$$

where f is a metric function and χ is the twist potential, that is non-zero for non-static metrics.

The metric components can be calculated from the Ernst potential as quadratures,

$$A_\rho = \frac{4\,\rho}{(\varepsilon+\bar\varepsilon)^2}\,\chi_z, \tag{3}$$

$$A_z = -\frac{4\,\rho}{(\varepsilon+\bar\varepsilon)^2}\,\chi_\rho, \tag{4}$$

$$\gamma_\rho = \frac{\rho}{(\varepsilon+\bar\varepsilon)^2}\,(\varepsilon_\rho\bar\varepsilon_\rho - \varepsilon_z\bar\varepsilon_z), \tag{5}$$

$$\gamma_z = \frac{\rho}{(\varepsilon+\bar\varepsilon)^2}\,(\varepsilon_\rho\bar\varepsilon_z + \varepsilon_z\bar\varepsilon_\rho), \tag{6}$$

$$ds^2 = -f(dt - Ad\phi)^2 + \frac{1}{f}\{e^{2\gamma}(d\rho^2 + dz^2) + \rho^2\,d\phi^2\}. \tag{7}$$

Although the Ernst equation is an integrable system, we are very far from analitically implementing physics at will (Schwarzschild, Kerr solutions). Therefore, we shall work with aproximate solutions. We shall solve the Ernst equation up to the seventh order in the pseudospherical radius, $r = \sqrt{\rho^2 + z^2}$, and the obtain the metric functions from it.

The results will be given in terms of invariant quantities. Concerning the probe, the existence of isometries leads to the appearance of two conserved quantities in geodesic motion, the energy and angular momentum per unit of mass of the test particle,

$$E = -\partial_t \cdot u = f\,(\dot t - A\dot\phi), \tag{8}$$

$$l = \partial_\phi \cdot u = f\,A\,(\dot t - A\dot\phi) + \frac{1}{f}\,r^2\,\dot\phi, \tag{9}$$

where the overhead dot stands for the derivative with respect to proper time and u is the velocity of the probe.

And considering the gravitational source, we have the Geroch-Hansen multipole moments, P_n [2,3]. In classical mechanics we would obtain the multipole moments as coefficients in the expansion of the gravitational potential in inverse powers of the radius and Legendre polynomials,

$$V(r,\theta) = -G\int d\vec{r}\,'\,\frac{\rho(\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|} = -G\sum_{n=0}^{\infty} P_n\,\frac{p_n(\cos\theta)}{r^{n+1}}, \tag{10}$$

which can be easily read from the expansion on the axis,

$$V(\rho=0) = -G\sum_{n=0}^{\infty}\frac{P_n}{z^{n+1}}. \tag{11}$$

In general relativity expressions are more involved, but still they can be obtained from the expansion of the Ernst potential in Weyl coordinates on the axis, [4]

$$\varepsilon(\rho=0) = \sum_{n=0}^{\infty}\frac{C_n}{z^{n+1}}, \tag{12}$$

$$P_n = C_n, \quad n \le 3, \tag{13}$$

$$P_4 = C_4 + \frac{1}{7}\,\bar{C}_0\,(C_1^2 - C_2\,C_0) \tag{14}$$

$$P_5 = C_5 + \frac{1}{3}\,\bar{C}_0\,(C_2\,C_1 - C_3\,C_0) + \frac{1}{21}\bar{C}_1\,(C_1^2 - C_2\,C_0), \ldots \tag{15}$$

although expressions remain simple only up to octupole moment. Since we have imposed an equatorial plane of symmetry, even multipole moments will be real (gravitational moments) and odd ones will be imaginary (rotational moments). The reason for expanding up to seventh order is precisely the aim of reaching the first non-linear terms.

The expressions for the metric components are rather lengthy. Details will be published elsewhere.[5]

3 Perihelion precession

For perihelion precession the geodesic equations for equatorial orbits will be written in terms of the constants of motion. The Binet equation for the orbits,

$${u_\phi}^2 = e^{-2\gamma}\left\{\frac{E^2 - f}{f^2\,(l - E\,A)^2} - u^2\right\} = F(u) - u^2, \tag{16}$$

is equivalent to a quasilinear equation,

$$u_{\phi\phi} = \frac{1}{2}\,F'(u) - u, \tag{17}$$

which can be solved perturbatively, but a small parameter is needed. A good candidate is $\varepsilon = P_0/l$, since from Kepler's laws, $l \sim \sqrt{P_0 r}$ in the far field region. To avoid secular terms a new coordinate,

$$\psi = \omega\phi, \tag{18}$$

$$\omega = \sqrt{1 + \sum \omega_i \epsilon^i}. \tag{19}$$

is introduced and redefined at every step of perturbation. We are lead to a hierarchy of of harmonic equations for the terms of the series,

$$u = \epsilon^2 \sum_{n=0}^{11} u_n \, \epsilon^n + O(\epsilon^{14}), \tag{20}$$

the first of which is Keplerian.

But we are more interested in the frequency ω, which will furnish the perihelion precession,

$$\begin{aligned} \Delta\phi = 2\,\pi\,(\omega^{-1} - 1) = \Delta_{class} + \Delta_{P_0} + \Delta_{P_1} + \Delta_{P_2} + \Delta_{P_3} + \\ + \Delta_{P_4} + \Delta_{P_5} + \Delta_{P_1-P_2} + \Delta_{P_1-P_3} + \Delta_{P_1-P_4} + \Delta_{P_2-P_3}, \end{aligned} \tag{21}$$

the terms of which we have classified here into classical and relativistic and these according to their multipole content.

Results are a bit cumbersome to produce here, but they can be summarized in the following way:

The Schwarschild term, Δ_{P_0}, as it is well known, is positive and the pure P_2 term is negative for positive quadrupole moment. This suggest a pattern of alternation of signs that has been checked up to the considered order of perturbation. For positive multipole moments, their linear contribution in $\Delta_{P_{4n}}$ is positive and negative in $\Delta_{P_{4n+2}}$. This means, for instance, that for an approximately ellipsoidal source, all gravitational moment contributions are positive, since the sign of the multipole moments also alternates in the same way. There is no difference of sign between classical and relativistic terms.

On the other hand, rotational terms for $P_{2n+1} = iJ_{2n+1}$ are sensitive to the orientation of the orbit of the probe particle. For a counterrotating orbit the linear dipole term is positive, whereas the linear octupole term is negative. Again, there is a pattern of alternation of signs, which is similar to the one found for gravitational moments, the contribution of the pure multipole is positive for $\Delta_{P_{4n+1}}$ and negative for $\Delta_{P_{4n-1}}$ for a counterrotating orbit.

Finally the coupling terms bear the same sign as the product of the respective linear multipole terms, that is, $\mathrm{sign}\,(\Delta_{P_i-P_j}) = \mathrm{sign}\,(\Delta_{P_i})\cdot\mathrm{sign}\,(\Delta_{P_j})$. This rule is also true for self-couplings. Therefore all quadratic terms are positive.

4 Line of nodes precession

In this section we shall derive the precession of the nodes of a slightly tilted geodesic with respect to a nearby geodesic circle on the equatorial plane.

Perturbing the geodesic equations we arrive at and equation,

$$\ddot{\delta}^\theta - \frac{1}{2}\, g^{\theta\theta}\, g_{ij,\theta\theta}\, \dot{x}^i\, \dot{x}^j\, \delta^\theta = 0, \tag{22}$$

$$\delta_{\phi\phi} + \Omega^2\, \delta = 0, \tag{23}$$

$$\Omega^2 = -\, \frac{1}{2\,\dot{\phi}^2}\, g^{\theta\theta}\, g_{\rho\sigma,\theta\theta}\Big|_{r=R,\theta=\pi/2} x^\rho\, \dot{x}^\sigma, \tag{24}$$

for $\delta = \delta^\theta$, the zeros of which will determine the nodes of the orbit.

This equation can be handled perturbatively to yield the amount of node precession,

$$\begin{aligned}\Delta\phi = \Delta_{class} + \Delta_{P_1} + \Delta_{P_2} + \Delta_{P_3} + \Delta_{P_4} + \Delta_{P_5} + \\ + \Delta_{P_1-P_2} + \Delta_{P_1-P_3} + \Delta_{P_1-P_4} + \Delta_{P_2-P_3},\end{aligned} \tag{25}$$

where there will be no contribution from the pure mass term, since for spherical distributions of mass there is no privileged plane of symmetry.

Concerning gravitational moments, the linear contribution of a positive quadrupole moment to Δ_{P_2} is positive. For higher multipoles the sign of the contributions alternates in the opposite way as for perihelion precession, that is, their linear contribution in $\Delta_{P_{4n}}$ is negative and positive in $\Delta_{P_{4n+2}}$.

A similar qualitative behaviour is found for rotational moments. The linear contribution of the dipole moment to Δ_{P_1} is negative for a counterrotating orbit. The pattern of signs is again the opposite to the one found for perihelion precession. The linear contribution of the multipole is negative for $\Delta_{P_{4n+1}}$ and positive for $\Delta_{P_{4n-1}}$ for a counterrotating orbit.

So far, the main qualitative difference between both cases would be an overall sign. However new features appear when couplings are considered.

Classical terms follow the same pattern as the one derived for perihelion precession, $\mathrm{sign}\,(\Delta_{P_i-P_j}) = \mathrm{sign}\,(\Delta_{P_i})\cdot\mathrm{sign}\,(\Delta_{P_j})$, whereas the relativistic ones follow the opposite one, $\mathrm{sign}\,(\Delta_{P_i-P_j}) = -\mathrm{sign}\,(\Delta_{P_i})\cdot\mathrm{sign}\,(\Delta_{P_j})$.

This behaviour has some curious consequences. Whereas classical quadratic terms increase node precession, relativistic terms diminish it.

5 Conclusions

I would be interesting to explore whether the patterns observed for perihelion and node precession are general. However, there is no general formula for arbitrary orders of the multipole moments in terms of the Ernst potential expansion on the axis and this may pose severe difficulties.

Other generalizations might include electromagnetic multipole moments and charged test particles or other precession phenomena, such as the dragging of gyroscopes.

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